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Summability of formal solution of Cauchy problem for some PDE with variable coefficients

By

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Abstract

We give a proof of the summability of formal solution for Cauchy problem of linear first order partial differential equations with respect to t and with t dependent coefficients under some global conditions for Cauchy data.

§ 1. Result

We consider the following Cauchy problem for linear partial differential equations of first order with respect to t and with t dependent coefficients

$$(1.1) \qquad \begin{cases} \partial_t u(t,x) = \displaystyle\sum_{(i,\alpha)}^{finite} a_{i\alpha} t^i \partial_x^\alpha u(t,x) \\ u(0,x) = \varphi(x) \in \mathcal{O}, \end{cases}$$

where $(t,x) \in \mathbb{C}^2$, $a_{i\alpha} \in \mathbb{C}$ and $\mathcal{O}$ denotes the set of holomorphic functions in a neighborhood of the origin.

The Cauchy problem (1.1) has a unique formal solution of the form

$$(1.2) \qquad \hat{u}(t,x) = \sum_{n\geq 0} u_n(x) \frac{t^n}{n!}.$$

We have an interest in the case where the formal solution is divergent. We assume that for the operator

$$(1.3) \qquad \max \alpha > 1,$$

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with $a_{i\alpha} \neq 0$. In this case, the formal solution (1.2) is not convergent in general.

We shall study the summability of the formal solution.

For the operator we assume that for indices (i, α) with $a_{i\alpha} \neq 0$, the number $\alpha/(i+1)$ is a constant, that is

$$\frac{\alpha}{i+1} = \frac{p}{q}, \tag{1.4}$$

where p and q are relatively prime. We call this number $\alpha/(i+1)$ the modified order of the operator (cf. M. Miyake [6]). In this case, since we see $\alpha = pj$ and $i = qj - 1$ for $j \geq 1$, our equation is rewritten as follows.

$$\partial_t u(t,x) = \sum_{j=1}^{\nu} a_j t^{qj-1} \partial_x^{pj} u(t,x), \tag{1.5}$$

where $\nu \geq 1$ and $a_\nu \neq 0$. Then the assumption (1.3) means

$$p\nu > 1.$$

In the case where $(p, q) = (2, 1)$ and $\nu = 1$, the equation (1.5) is the heat equation.

Before stating our result, we give some notations and definitions in Ramis or Balser ways (cf. W. Balser [1]).

For $d \in \mathbb{R}$, $\beta > 0$ and $\rho\,(0 < \rho \leq \infty)$, we define a sector $S = S(d, \beta, \rho)$ by

$$S(d, \beta, \rho) := \left\{ t \in \mathbb{C}; |d - \arg t| < \frac{\beta}{2}, 0 < |t| < \rho \right\}, \tag{1.6}$$

where d, β and ρ are called the direction, the opening angle and the radius of S, respectively. We write $S(d, \beta, \infty) = S(d, \beta)$ for short.

Let $k > 0$, $S = S(d, \beta)$ and $B(r) := \{x \in \mathbb{C}; |x| \leq r\}$. Let $v(t,x) \in \mathcal{O}(S \times B(r))$ which means that $v(t,x)$ is holomorphic in $S \times B(r)$. Then we define that $v(t,x) \in \mathrm{Exp}_t(k, S \times B(r))$, if for any closed subsector S' of S, there exist some positive constants C and δ such that

$$\max_{|x| \leq r} |v(t,x)| \leq C e^{\delta |t|^k}, \quad t \in S'. \tag{1.7}$$

For $k > 0$, we define that $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x) t^n / n! \in \mathcal{O}[[t]]_{1/k}$ ($\hat{v}(t,x)$ is a formal power series of Gevrey order $1/k$), if $v_n(x)$ are holomorphic on a common closed disk $B(r)$ for some $r > 0$ and there exist some positive constants C and K such that for any n,

$$\max_{|x| \leq r} |v_n(x)| \leq C K^n \Gamma\left(1 + \left(1 + \frac{1}{k}\right) n\right). \tag{1.8}$$

Here Γ denotes the gamma function.

Let $k > 0$, $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x)t^n/n! \in \mathcal{O}[[t]]_{1/k}$ and $v(t,x)$ be an analytic function on $S(d,\beta,\rho) \times B(r)$. Then we define that

$$v(t,x) \cong_k \hat{v}(t,x) \quad \text{in } S = S(d,\beta,\rho), \tag{1.9}$$

if for any closed subsector S' of S, there exist some positive constants C and K such that for any N, we have

$$\max_{|x|\le r} \left| v(t,x) - \sum_{n=0}^{N-1} v_n(x) \frac{t^n}{n!} \right| \le CK^N |t|^N \Gamma\left(1 + \left(1 + \frac{1}{k}\right) N\right), \quad t \in S'. \tag{1.10}$$

For $k > 0$, $d \in \mathbb{R}$ and $\hat{v}(t,x) \in \mathcal{O}[[t]]_{1/k}$, we define that $\hat{v}(t,x)$ is *k-summable* in d direction ($\hat{v}(t,x) \in \mathcal{O}\{t\}_{k,d}$) if there exist a sector $S = S(d,\beta,\rho)$ with $\beta > \pi/k$ and an analytic function $v(t,x)$ on $S \times B(r)$ such that $v(t,x) \cong_k \hat{v}(t,x)$ in S.

We remark that the function $v(t,x)$ above for a *k-summable* $\hat{v}(t,x)$ is unique if it exists. Therefore such a function $v(t,x)$ is called the *k-sum* of $\hat{v}(t,x)$ in d direction.

Under the above preparations, our result is stated as follows.

Theorem 1.1. *For a fixed $d \in \mathbb{R}$, we define $d_\ell = (q/p)d + (\arg a_\nu + 2\pi\ell)/p\nu$ for $\ell = 0, 1, \cdots, p\nu - 1$. Let*

$$k = \frac{q\nu}{p\nu - 1} \tag{1.11}$$

and for some $\varepsilon > 0$ and $r > 0$,

$$\Omega_x(p,\nu) := \cup_{\ell=0}^{p\nu-1} S(d_\ell, \varepsilon) \bigcup B(r). \tag{1.12}$$

We assume that

$$\varphi(x) \in \mathrm{Exp}_x\left(\frac{p\nu}{p\nu - 1}, \Omega_x(p,\nu)\right). \tag{1.13}$$

Then the formal solution $\hat{u}(t,x)$, which is given by (1.2), of the Cauchy problem (1.1) is k-summable in a direction d.

§ 2. Gevrey order of formal solution

We recall our Cauchy problem

$$\begin{cases} \partial_t u(t,x) = \sum_{j=1}^{\nu} a_j t^{qj-1} \partial_x^{pj} u(t,x), \\ u(0,x) = \varphi(x). \end{cases} \tag{2.1}$$

Gevrey order of formal solution of Cauchy problem (2.1) is given by the following proposition.

Proposition 2.1. *Let $\hat{u}(t,x) = \sum_{n=0}^{\infty} u_n(x)t^n/n!$ be the formal solution of the Cauchy problem (2.1). Then we have*

$$\hat{u}(t,x) \in \mathcal{O}[[t]]_{1/k} \quad \left(k = \frac{q\nu}{p\nu - 1}\right). \tag{2.2}$$

This proposition follows from the results which are proved by many mathematicians (cf. M. Miyake and Y. Hashimoto [7] and A. Shirai [9] and their references). But we give the proof, because the notations which are represented here will be employed later.

Proof. The coefficients $u_n(x)$ of the formal solution satisfy the following recurrence formula: $u_0(x) = \varphi(x)$ and for $n \geq 0$,

$$u_{n+1}(x) = \sum_{j=1}^{\nu} a_j \cdot [n]_{qj-1} \cdot u_{n-qj+1}^{(pj)}(x), \tag{2.3}$$

where $u_{-n}(x) \equiv 0$ $(n > 0)$ and the notation $[n]_\ell$ is defined by

$$[n]_\ell = \begin{cases} n(n-1)\cdots(n-\ell+1), & \ell \geq 1 \\ 1, & \ell = 0. \end{cases} \tag{2.4}$$

From the construction of recurrence formula (2.3), we can put

$$u_{qn}(x) = A(n)\varphi^{(pn)}(x) \quad (n \geq 0), \tag{2.5}$$

and $u_\ell(x) \equiv 0$ for $\ell \neq qn$ $(n \geq 0)$. Then we obtain the recurrence formula of $A(n)$: For $n \geq 0$,

$$A(n+1) = \sum_{j=1}^{\nu} a_j \cdot [qn + q - 1]_{qj-1} \cdot A(n - j + 1), \tag{2.6}$$

where $A(0) = 1$ and we put $A(-\ell) = 0$ for $\ell > 0$.

We put

$$\hat{f}(t) := \sum_{n \geq 0} A(n)t^n, \tag{2.7}$$

which is the generating function of $A(n)$. Then we obtain the following Gevrey estimate of $\hat{f}$.

Lemma 2.2. *Let*

$$\tilde{k} = \frac{\nu}{q\nu - 1}. \tag{2.8}$$

Then we have

$$\hat{f}(t) \in \mathbb{C}[[t]]_{1/\tilde{k}}. \tag{2.9}$$

Proposition 2.1 follows from Lemma 2.2 immediately. In fact, we have for some $r > 0$

$$\begin{aligned}\max_{|x|\le r}|u_n(x)| &= \max_{|x|\le r}\left|A(n/q)\varphi^{(pn/q)}(x)\right| \\ &\le C_1K_1^n\Gamma(1+n/(\tilde{k}q))\left(\frac{p}{q}n\right)! \\ &\le C_2K_2^n\Gamma\left(1+\left(\frac{1}{\tilde{k}q}+\frac{p}{q}\right)n\right)\end{aligned}$$

and

$$\frac{1}{\tilde{k}q}+\frac{p}{q}=\frac{1+\tilde{k}p}{\tilde{k}q}=\frac{q\nu-1+p\nu}{q\nu}=1+\frac{p\nu-1}{q\nu}=1+\frac{1}{k},$$

where $A(n/q)=0$ if $n/q\notin\mathbb{N}_0=\{0,1,2,\cdots\}$ and C_i and K_i are some positive constants for $i=1,2$. This implies the desired Gevrey estimate of the formal solution $\hat{u}$. $\square$

⟨*Proof of Lemma 2.2.*⟩ We put $B(n):=A(n)/n!^{1/\tilde{k}}\Leftrightarrow A(n)=B(n)n!^{1/\tilde{k}}$ and divide both hand sides of (2.6) by $(n+1)!^{1/\tilde{k}}$.

$$B(n+1)=\sum_{j=1}^{\nu}a_j\cdot[qn+q-1]_{qj-1}\left(\frac{(n-j+1)!}{(n+1)!}\right)^{1/\tilde{k}}B(n-j+1). \tag{2.10}$$

Here we put $I_j(n;q):=[qn+q-1]_{qj-1}\left(\frac{(n-j+1)!}{(n+1)!}\right)^{1/\tilde{k}}$. Then we have

Lemma 2.3.

$$I_j(n;q)\le q^{qj-1}. \tag{2.11}$$

By admitting this lemma for a while, we continue the proof of Lemma 2.2.

Now, we consider the following recurrence formula of $C(n)$; $C(0)=1$ and for $n\ge 0$,

$$C(n+1)=\sum_{j=1}^{\nu}|a_j|q^{qj-1}C(n-j+1), \tag{2.12}$$

where $C(-\ell)=0$ if $\ell>0$. We see that $C(n)\ge|B(n)|=|A(n)|/n!^{1/\tilde{k}}$ from the construction of the recurrence formulas. By putting

$$h(t)=\sum_{n\ge 0}C(n)t^n,$$

we get $h(t)-1=\sum_{j=1}^{\nu}\{|a_j|/q\}(q^qt)^jh(t)$, that is

$$h(t)=\frac{1}{1-\sum_{j=1}^{\nu}\{|a_j|/q\}(q^qt)^j},$$

which is convergent in a neighborhood of the origin. Therefore there exist positive constants C_3 and K_3 such that $C(n) \le C_3 K_3^n$ for any n, and by using Stirling's formula for the Gamma function, we obtain the desired Gevrey estimate for $A(n)$.

Finally, we prove Lemma 2.3.

⟨*Proof of Lemma 2.3.*⟩ When $q = 1$ and $j = 1$, we have $[qn + q - 1]_{qj-1} = 1$. Therefore we get

$$I_1(n;1) = \left(\frac{n!}{(n+1)!}\right)^{1/\tilde{k}} = \left(\frac{1}{n+1}\right)^{1/\tilde{k}} < 1.$$

In other case, we have

$$\begin{aligned}
&I_j(n;q)\\
&= (qn+q-1)(qn+q-2)\cdots(qn+q-(qj-1))\left(\frac{1}{(n+1)n\cdots(n-j+2)}\right)^{1/\tilde{k}}\\
&= q^{qj-1}\left(n+1-\frac{1}{q}\right)\left(n+1-\frac{2}{q}\right)\cdots\left(n-j+1+\frac{1}{q}\right)\left(\frac{1}{(n+1)n\cdots(n-j+2)}\right)^{\frac{q\nu-1}{\nu}}\\
&= q^{qj-1}\left\{\frac{\left(n+1-\frac{1}{q}\right)^{\nu}\left(n+1-\frac{2}{q}\right)^{\nu}\cdots\left(n-j+1+\frac{1}{q}\right)^{\nu}}{(n+1)^{q\nu-1}n^{q\nu-1}\cdots(n-j+2)^{q\nu-1}}\right\}^{1/\nu}\\
&= q^{qj-1}\left\{\frac{\left(n+1-\frac{1}{q}\right)^{\nu}\left(n+1-\frac{2}{q}\right)^{\nu}\cdots\left(n+1-\frac{q}{q}\right)^{\nu}}{(n+1)^{q\nu-1}\cdot n}\right\}^{1/\nu}\\
&\quad\times\left\{\frac{\left(n-\frac{1}{q}\right)^{\nu}\left(n-\frac{2}{q}\right)^{\nu}\cdots\left(n-\frac{q}{q}\right)^{\nu}}{n^{q\nu-2}\cdot(n-1)^2}\right\}^{1/\nu}\cdot\left\{\frac{\left(n-1-\frac{1}{q}\right)^{\nu}\cdots\left(n-1-\frac{q}{q}\right)^{\nu}}{(n-1)^{q\nu-3}\cdot(n-2)^3}\right\}^{1/\nu}
\end{aligned}$$

$$\begin{aligned}
&\cdots\times\left\{\frac{\left(n-j+3-\frac{1}{q}\right)^{\nu}\cdots(n-j+2)^{\nu}}{(n-j+1)^{q\nu-(j-1)}\cdot(n-j+2)^{j-1}}\right\}^{1/\nu}\\
&\times\left\{\frac{\overbrace{\left(n-j+2-\frac{1}{q}\right)^{\nu}\cdots\left(n-j+1+\frac{1}{q}\right)^{\nu}}^{q\nu-\nu}}{(n-j+2)^{q\nu-j}}\right\}^{1/\nu} < q^{qj-1}.
\end{aligned}$$

$\square$

§ 3. Summability of formal solution

Let $k = q\nu/(p\nu - 1)$. From Proposition 2.1 we see that

$$u_B(s,x) := (\hat{\mathcal{B}}_k \hat{u})(s,x) = \sum_{n\geq 0} \frac{u_n(x)}{\Gamma(1+n/k)} \frac{s^n}{n!}$$

is convergent in a neighborhood of $(s,x) = (0,0)$. In order to prove Theorem 1.1, we use the important lemma for the summability theory (cf. [1], D. Lutz, M. Miyake and R. Schäfke [5]).

Lemma 3.1. *Let $k > 0$, $d \in \mathbb{R}$ and $\hat{v}(t,x) = \sum v_n(x)t^n/n! \in \mathcal{O}[[t]]_{1/k}$. Then the following statements are equivalent:*

i) $\hat{v}(t,x) \in \mathcal{O}\{t\}_{k,d}$.

ii) $v_B(s,x) \in \mathrm{Exp}_s(k, S(d,\varepsilon) \times B(r))$ *for some* $\varepsilon > 0$ *and* $r > 0$.

For our purpose, we prepare a lemma for the summability of $\hat{f}(t)$, which will be proved in the next section.

Lemma 3.2. *Let $\tilde{k} = \nu/(q\nu - 1)$. Then we have $\hat{f}(t) = \sum_{n\geq 0} A(n)t^n \in \mathbb{C}\{t\}_{\tilde{k},d}$, where*

$$d \neq e_\ell \ (\ell = 0,1,\cdots,\nu-1) \quad \text{and} \quad e_\ell := -(\arg a_\nu + 2\pi\ell)/\nu.$$

Remark. $f_B(t) = (\hat{\mathcal{B}}_{\tilde{k}} \hat{f})(t)$ has ν singular points which are given by

$$\lambda_\ell = c_f a_\nu^{-1/\nu} \omega_\nu^{-\ell}, \quad \ell = 0,1,\cdots,\nu-1, \tag{3.1}$$

where $c_f = (q\tilde{k})^{(1-q\nu)/\nu}$ and $\omega_\nu = e^{2\pi i/\nu}$. Moreover, we have

$$f_B(t) \in \mathrm{Exp}_t(\tilde{k}, S(d,\varepsilon_0)), \tag{3.2}$$

where $d \neq e_\ell$ and $\varepsilon_0 > 0$.

Now, we put

$$(M^w \hat{f})(t) := \sum_{n\geq 0} A(n)w(n)t^n, \quad w(n) = \frac{(pn)!}{(qn)!}, \tag{3.3}$$

which is called the moment series of $\hat{f}$ with respect to weight function $w(n)$. Then we remark that $(M^w \hat{f})$ is a formal power series of Gevrey order $1/k_*$ where

$$\frac{1}{k_*} := \frac{1}{\tilde{k}} - (q-p) = \frac{q\nu - 1 - (q-p)\nu}{\nu} = \frac{p\nu - 1}{\nu} (> 0).$$

We give the following theorem for the summability of $(M^w \hat{f})(t)$.

Theorem 3.3. *Let $k_* = \nu/(p\nu - 1)$. Then we have $(M^w \hat{f})(t) \in \mathbb{C}\{t\}_{k_*,d}$, where*

$$d \neq e_\ell \ (\ell = 0, 1, \cdots, \nu - 1).$$

Remark.

$$(M^w f)_B(t) := (\hat{\mathcal{B}}_{k_*}(M^w \hat{f}))(t) = \sum_{n \geq 0} \frac{A(n)}{\Gamma(1 + n/k_*)} \frac{(pn)!}{(qn)!} t^n \tag{3.4}$$

has ν singular points which are given by

$$\lambda_{*\ell} = c_g a_\nu^{-1/\nu} \omega_\nu^{-\ell}, \quad \ell = 0, 1, \cdots, \nu - 1, \tag{3.5}$$

where $c_g = q^{1/\nu} p^{-p} {k_*}^{(1-p\nu)/\nu}$. Moreover, we have

$$(M^w f)_B(t) \in \mathrm{Exp}_t(k_*, S(d, \varepsilon_0)), \tag{3.6}$$

where $d \neq e_\ell$ and $\varepsilon_0 > 0$.

The directions for which some divergent series is not summable are called the singular directions of the divergent series. From the above facts, we have the following result for the moment series.

Remark. The set of singular directions of $(M^w \hat{f})$ coincides with the one of $\hat{f}$.

We give the proof of Theorem 1.1 by using Lemma 3.1 ii) and Theorem 3.3.

Proof. We recall

$$k = \frac{q\nu}{p\nu - 1}, \quad k_* = \frac{k}{q} = \frac{\nu}{p\nu - 1}.$$

Let $|s|$ and $|x|$ be sufficiently small. By Cauchy's integral formula for u_B, we have

$$\begin{aligned} u_B(s, x) &= \sum_{n \geq 0} \frac{u_n(x)}{\Gamma(1 + n/k)} \frac{s^n}{n!} = \sum_{n \geq 0} \frac{u_{qn}(x)}{\Gamma(1 + qn/k)} \frac{s^{qn}}{(qn)!} \\ &= \sum_{n \geq 0} \frac{A(n)\varphi^{(pn)}(x)}{\Gamma(1 + n/k_*)} \frac{s^{qn}}{(qn)!} \\ &= \frac{1}{2\pi i} \oint_{|\zeta| = r_0} \frac{\varphi(x + \zeta)}{\zeta} \sum_{n \geq 0} \frac{A(n)}{\Gamma(1 + n/k_*)} \frac{(pn)!}{(qn)!} \left(\frac{s^q}{\zeta^p} \right)^n d\zeta \\ &= \frac{1}{2\pi i} \oint_{|\zeta| = r_0} \frac{\varphi(x + \zeta)}{\zeta} (M^w f)_B \left(\frac{s^q}{\zeta^p} \right) d\zeta, \end{aligned}$$

with $(|s|^q/R_*)^{1/p} < |\zeta| = r_0$, $R_* = |\lambda_{*\ell}|$ and $\lambda_{*\ell} = \mathit{constant} \times a_\nu^{-1/\nu} \omega_\nu^{-\ell}$.

We remark that $(M^w f)_B(s^q/\zeta^p)$ has ν singular points in $t(= s^q/\zeta^p)$ plane. Therefore we see that $(M^w f)_B(s^q/\zeta^p)$ has $p\nu$ singular points in ζ plane and the singular points are given by

$$\zeta_{\ell,m} := \left(\frac{s^q}{\lambda_{*\ell}}\right)^{1/p} \omega_p^m = constant \times s^{q/p} a_\nu^{1/p\nu} \omega_\nu^{\ell/p} \omega_p^m \tag{3.7}$$

for $\ell = 0, 1, \cdots, \nu - 1$ and $m = 0, 1, \cdots, p - 1$. For $n = 0, 1, \ldots, p\nu - 1$, we put

$$\zeta_n := constant \times s^{q/p} a_\nu^{1/p\nu} \omega_{p\nu}^n$$

and

$$d_n := \arg \zeta_n = \frac{q}{p} d + \frac{\arg a_\nu + 2\pi n}{p\nu} \tag{3.8}$$

for a fixed s with $\arg s = d$.

We consider the situation that $|s|$ becomes bigger along $\arg s = d$. In this case, we spit the path of integral into $p\nu$ arcs γ_n and $p\nu$ arcs Γ_n $(n = 0, 1, \cdots, p\nu - 1)$, where each γ_n consists of the arc between points of argument $d_n - \varepsilon/3$ and $d_n + \varepsilon/3$, and each Γ_n consists of the arc between points of argument $d_n + \varepsilon/3$ and $d_{n+1} - \varepsilon/3$ with $d_{p\nu} = d_0$. Since $\varphi(x)$ is analytic in $\Omega_x(p, \nu)$, we may deform γ_n into paths γ_{n,R_s} along the ray $\arg \zeta = d_n - \varepsilon/3$ to a point with modulus $R_s = c|s|^{q/p} + 1$ (c is some constant), then along the circle $|\zeta| = R_s$ to the ray $\arg \zeta = d_n + \varepsilon/3$ and back along this ray to the original circle. So we have

$$u_B(s, x) = \sum_{n=0}^{p\nu-1} \frac{1}{2\pi i} \left\{ \int_{\Gamma_n} + \int_{\gamma_{n,R_s}} \right\} \frac{\varphi(x + \zeta)}{\zeta} (M^w f)_B \left(\frac{s^q}{\zeta^p}\right) d\zeta. \tag{3.9}$$

In this expression, from the assumption that $\varphi(x) \in \mathrm{Exp}_x(p\nu/p\nu - 1, \Omega_x(p, \nu))$ and the fact that $(M^w f)_B(t) \in \mathrm{Exp}_t(k_*, S(d, \varepsilon_0))$ with $d \neq e_\ell$ as in Remark of Theorem 3.3, we can obtain the property of the analytic continuation and the desired exponential growth estimate of $u_B(s, x)$ in $S(d, \tilde{\varepsilon}) \times B(r)$ for some $\tilde{\varepsilon}$ and r. $\square$

§ 4. Proof of Lemma 3.2 and Theorem 3.3

We shall prove Lemma 3.2 and Theorem 3.3, which are the results of k-summability of the formal power series $\hat{f}$ and the moment series $M^w \hat{f}$. For the purpose, we shall give the results for k-summability of the formal solution of a linear ordinary differential equation in subsection 4.1. By applying their results to $\hat{f}$ and the moment series $M^w \hat{f}$, we shall give the proof of Lemma 3.2 and Theorem 3.3 in subsection 4.2.

§ 4.1. Summability of formal solution for an ODE

In this subsection, we only give results for k-summability of the formal solution of a linear ordinary differential equation. For the proof, see B. L. J. Braaksma [2], [3], S. Ōuchi [8] and K. Ichinobe [4].

Let k be positive rational number and let us consider the following polynomials.

$$P_0(\zeta) = \sum_{j=0}^{m_1} A_j \zeta^j, \quad P_1(t,\eta) = \sum_{j=0}^{m_1+m_2} a_j(t)\eta^j,$$

where $m_1 \in \mathbb{N} = \{1, 2, \cdots\}$, $m_2 \in \mathbb{N}_0$, $A_j, \zeta, \eta \in \mathbb{C}$ and $a_j(t) \in \mathcal{O}$.

Here we assume that $m_1 k \in \mathbb{N}$, $A_0 \neq 0$, $A_{m_1} \neq 0$ and if $jk \notin \mathbb{N}_0$, then $A_j = 0$, and

$$O(a_j) \begin{cases} \geq 0, & 0 \leq j \leq m_2 - 1, \\ > (j - m_2)k, & m_2 \leq j \leq m_1 + m_2, \end{cases}$$

where $O(a)$ denotes the order of zeros of a function $a(t)$ at $t = 0$.

We consider the following linear ordinary differential equation.

$$P_0(t^k \delta_t)\delta_t^{m_2} y = g(t) + P_1(t, \delta_t)y, \tag{4.1}$$

where $\delta_t = t(d/dt)$ is the Euler operator and $g(t) \in \mathcal{O}$.

Here, we give the definition of the Newton polygon for the equation (4.1) (cf. M. Miyake and Y. Hashimoto [7] and A. Shirai [9]).

Let $L = \sum_{i,j}^{\text{finite}} \ell_{i,j} t^i \delta_t^j$ be a differential operator. We define a domain $\mathrm{N}(i, j)$ by

$$\mathrm{N}(i,j) := \{(x,y) \in \mathbb{R}^2; x \leq j,\ y \geq i\} \text{ for } \ell_{i,j} \neq 0,$$

and $\mathrm{N}(i, j) := \phi$ for $\ell_{i,j} = 0$. Then the Newton polygon $\mathrm{N}(L)$ of the operator L is defined by

$$\mathrm{N}(L) := \mathrm{Ch}\left\{ \bigcup_{i,j}^{\text{finite}} \mathrm{N}(i,j) \right\},$$

where $\mathrm{Ch}\{\cdots\}$ denotes the convex hull of points in $\cup_{i,j}\mathrm{N}(i, j)$. By employing this definition, we define the Newton polygon for the equation (4.1) by

$$\mathrm{N}\left(P_0(t^k\delta_t)\delta_t^{m_2} - P_1(t,\delta_t)\right).$$

The above assumptions means

$$\mathrm{N}(P_0(t^k\delta_t)\delta_t^{m_2}) = \mathrm{N}\left(P_0(t^k\delta_t)\delta_t^{m_2} - P_1(t,\delta_t)\right) \quad \text{and} \quad \mathrm{N}(P_0(t^k\delta_t)\delta_t^{m_2}) \supset \mathrm{N}(P_1(t,\delta_t)).$$

In this sense, $P_0(t^k\delta_t)\delta_t^{m_2}$ is called the principal operator of the equation (4.1).

In order to state the result of k-summability of the formal solution for (4.1), we define a characteristic equation associated with the principal operator $P_0(t^k\delta_t)\delta_t^{m_2}$

$$P_0(\lambda) = 0. \tag{4.2}$$

Let λ_i $(i = 1, 2, \dots, m_1)$ be the roots of (4.2) and we put

$$\theta_{i,n} := \frac{\arg \lambda_i + 2\pi n}{k}, \quad n = 0, 1, \dots, k-1. \tag{4.3}$$

Then we have

Theorem 4.1. *Let* $\hat{y}(t) = \sum_{n=0}^{\infty} y_n t^n$ *be a formal power series solution of (4.1). Then* $\hat{y}(t) \in \mathbb{C}\{t\}_{k,d}$, *where*

$$d \neq \theta_{i,n} \quad (i = 1, 2, \dots, m_1,\ n = 0, 1, \dots, k-1). \tag{4.4}$$

Corollary 4.2. *Let* $y_B(s) := (\hat{\mathcal{B}}_k \hat{y})(s) = \sum_{n=0}^{\infty} y_n s^n / \Gamma(1 + n/k)$. *Then* $y_B(s)$ *has* $m_1 k$ *singular points in* s *complex plane at roots of*

$$P_0(ks^k) = 0, \tag{4.5}$$

which is called a singular equation of $y_B(s)$. *Moreover, we have*

$$y_B(s) \in \mathrm{Exp}_s(k; S(d, \varepsilon_0)), \tag{4.6}$$

for some $\varepsilon_0 > 0$, *where* $d \neq \theta_{i,n} (i = 1, 2, \dots, m_1,\ n = 0, 1, \dots, k-1)$.

§ 4.2. Proof of Lemma 3.2 and Theorem 3.3

⟨*Proof of Lemma 3.2.*⟩ In order to prove Lemma 3.2, it is enough to seek the differential equation which is satisfied by $\hat{f}$. We recall that the coefficients $A(n)$ satisfy the following recurrence formula

$$A(n+1) = \sum_{j=1}^{\nu} a_j \cdot [qn + q - 1]_{qj-1} \cdot A(n - j + 1), \tag{4.7}$$

where $A(0) = 1$ and $A(-\ell) = 0$ for $\ell > 0$.

By multiplying both hand sides of (4.7) by t^{n+1} and taking sum of $n \geq 0$, we get

$$\begin{aligned}\hat{f}(t) - 1 &= \sum_{j=1}^{\nu} a_j t^j \sum_{n \geq 0} [qn + q - 1]_{qj-1} A(n - j + 1) t^{n-j+1} \\ &= \sum_{j=1}^{\nu} a_j t^j \sum_{n \geq 0} [qn + qj - 1]_{qj-1} A(n) t^n.\end{aligned}$$

Here we use the another notation.

$$[qn+qj-1]_{qj-1} = (qn+qj-1)(qn+qj-2)\cdots(qn+qj-(qj-1)) =: (qn+1)_{qj-1},$$

where the notation $(n)_k$ denotes the Pochhammer symbol. We obtain the differential equation which is satisfied by $\hat{f}$:

$$\hat{f}(t) - 1 = \sum_{j=1}^{\nu} a_j t^j (q\delta_t + 1)_{qj-1} \hat{f}(t). \tag{4.8}$$

From the results in the previous subsection, we notice that the principal operator of (4.8) is given by

$$P_0 = a_\nu t^\nu (q\delta_t)^{q\nu-1} - 1 \tag{4.9}$$

and a singular equation of $f_B(t)$ is given by

$$a_\nu t^\nu (q\tilde{k})^{q\nu-1} - 1 = 0. \tag{4.10}$$

Therefore the singular points of $f_B(t)$, which are the roots of (4.10), are given by

$$t = \lambda_\ell = c_f a_\nu^{-1/\nu} \omega_\nu^{-\ell}, \quad \ell = 0, 1, \cdots, \nu - 1, \tag{4.11}$$

where $c_f = (q\tilde{k})^{(1-q\nu)/\nu}$.

Hence from Theorem 4.1, we obtain that $\hat{f}$ is $\tilde{k}$-summable in a direction d, where

$$d \neq e_\ell = \arg \lambda_\ell = -\frac{\arg a_\nu + 2\pi\ell}{\nu}, \quad \ell = 0, 1, \cdots, \nu - 1. \tag{4.12}$$

□

⟨*Proof of Theorem 3.3.*⟩ We prove Theorem 3.3 in the similar way to proof of Lemma 3.2. We seek the differential equation for $M^w \hat{f}$.

We notice

$$w(n+1) = \frac{(pn+1)_p}{(qn+1)_q} w(n) = \cdots = \frac{(p(n-j+1)+1)_{pj}}{(q(n-j+1)+1)_{qj}} w(n-j+1).$$

By multiplying both hand sides of (4.7) by $w(n+1)t^{n+1}$ and taking sum of $n \geq 0$, we get

$$\begin{aligned}
&(M^w \hat{f})(t) - 1 \\
&= \sum_{j=1}^{\nu} a_j t^j \sum_{n\geq 0} [qn+q-1]_{qj-1} A(n-j+1) \frac{(p(n-j+1)+1)_{pj}}{(q(n-j+1)+1)_{qj}} w(n-j+1) t^{n-j+1} \\
&= \sum_{j=1}^{\nu} a_j t^j \sum_{n\geq 0} (qn+1)_{qj-1} \frac{(pn+1)_{pj}}{(qn+1)_{qj}} A(n) w(n) t^n \\
&= \frac{1}{q} \sum_{j=1}^{\nu} a_j t^j \sum_{n\geq 0} \frac{(pn+1)_{pj}}{n+j} A(n) w(n) t^n \\
&= \frac{1}{q} \sum a_j D_t^{-1} t^{j-1} (p\delta_t + 1)_{pj} (M^w \hat{f})(t),
\end{aligned}$$

where $D_t^{-1} = \int_0^t$. Therefore we have

$$\frac{d}{dt}(M^w \hat{f})(t) = \sum_{j=1}^{\nu} \frac{a_j}{q} t^{j-1} (p\delta_t + 1)_{pj} (M^w \hat{f})(t)$$

$$\iff \delta_t (M^w \hat{f})(t) = \sum_{j=1}^{\nu} \frac{a_j}{q} t^j (p\delta_t + 1)_{pj} (M^w \hat{f})(t). \tag{4.13}$$

We notice that the principal operator of (4.13) is given by

$$Q_0 = \frac{a_\nu}{q} t^\nu (p\delta_t)^{p\nu} - \delta_t = \left(\frac{p}{q} a_\nu t^\nu (p\delta_t)^{p\nu-1} - 1\right)\delta_t \tag{4.14}$$

and a singular equation of $(M^w f)_B(t)$ is given by

$$\frac{p}{q} a_\nu t^\nu (pk_*)^{p\nu-1} - 1 = 0. \tag{4.15}$$

Therefore the singular points of $M^w \hat{f}(t)$, which are the roots of (4.15), are given by

$$t = \lambda_{*\ell} = c_g a_\nu^{-1/\nu} \omega_\nu^{-\ell}, \quad \ell = 0, 1, \cdots, \nu - 1, \tag{4.16}$$

where $c_g = q^{1/\nu} p^{-p} {k_*}^{(1-p\nu)/\nu}$. Hence we obtain that $M^w \hat{f}$ is k_*-summable in a direction d, where $d \neq e_\ell$ $(\ell = 0, 1, \ldots, \nu - 1)$. $\square$

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